Disjoint dynamics on weighted Orlicz spaces

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- X: a complex Banach space.
- $T : X \to X$ a bounded linear operator.
- T is **topologically transitive** if for any nonempty open subsets U and V of X, $T^n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$.
- T is **topologically mixing** if for any nonempty open subsets U and V of X, $T^n(U) \cap V \neq \emptyset$ from some n onwards.

Disjoint topological transitivity

Definition (Bès and Peris, JMAA 2007)

Given $N \geq 2$, the operators $T_1, T_2, \cdots, T_N$ on a Banach space X are *disjoint topologically transitive* if given nonempty open sets $U, V_1, \cdots, V_N \subset X$, there is some $n \in \mathbb{N}$ such that

$$\emptyset \neq U \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \cdots \cap T_N^{-n}(V_N).$$

If the above condition is satisfied from some n onwards, then $T_1, T_2, \cdots, T_N$ are called *d-mixing*.

Theorem (Bès and Peris, JMAA 2007)

Let $1 \leq p < \infty$*. For* $l = 1, 2, \ldots, N$*, let* $(w_{l,j})_{j \in \mathbb{Z}}$ *be a bounded sequence of positive numbers, and let* B_l *be the associated backward shift on* $\ell^p(\mathbb{Z})$ *given by* $B_l e_j := w_{l,j} e_{j-1}$*. For any integers* $1 \leq r_1 < r_2 < \ldots < r_N$*, the following conditions are equivalent.*

(i) $B_1^{r_1}, B_2^{r_2}, \ldots, B_N^{r_N}$ *are disjoint topologically transitive.*

(ii) *Given* $\varepsilon > 0$ *and* $q \in \mathbb{N}$*, there exists* $m \in \mathbb{N}$ *so that for all* $|j| \leq q$*, we have:*

Theorem (continued)

(a) *if* $1 \leq l \leq N$,

$$\prod_{i=j-r_l m+1}^{j} w_{l,i} < \varepsilon \quad \text{and} \quad \prod_{i=j+1}^{j+r_l m} w_{l,i} > \frac{1}{\varepsilon}.$$

(b) *if* $1 \leq s < l \leq N$,

$$\frac{\displaystyle\prod_{i=j+(r_l-r_s)m+1}^{j+r_l m} w_{s,i}}{\displaystyle\prod_{i=j+1}^{j+r_l m} w_{l,i}} < \varepsilon \quad \text{and} \quad \frac{\displaystyle\prod_{i=j-(r_l-r_s)m+1}^{j+r_s m} w_{l,i}}{\displaystyle\prod_{i=j+1}^{j+r_s m} w_{s,i}} < \varepsilon.$$

- The bilateral shift on $\ell^p(\mathbb{Z})$

- The translation operator on the Lebesgue space $L^p(G)$ of locally compact groups G

- G: a locally compact group with identity e.
- λ: a right Haar measure of G.
- $L^p(G)$: the complex Lebesgue space of G w.r.t. λ.
- σ: a complex Borel measure on G.

- The convolution operator $T_\sigma : L^p(G) \to L^p(G)$ is defined by

$$T_\sigma(f) = f * \sigma \qquad (f \in L^p(G))$$

where the convolution

$$f * \sigma(x) = \int_G f(xy^{-1}) d\sigma(y)$$

exists λ-almost everywhere.

- If $\sigma = \delta_a$ which is a unit point mass at $a \in G$, then

$$(T_\sigma f)(x) = \int_{\mathbb{R}} f(xy^{-1}) d\delta_a(y) = f(xa^{-1})$$

which is called a translation operator.

A function $w : G \to (0, \infty)$ is called a weight on G.

One can define a weighted translation operator by

$$\begin{aligned} T_{\delta_a,w} f(x) &= w(x)(f * \delta_a)(x) \\ &= w(x) \int_G f(xy^{-1}) d\delta_a(y) \\ &= w(x) f(xa^{-1}) \qquad (x \in G). \end{aligned}$$

Example

Let $G = \mathbb{Z}$. Then the bilateral forward weighted shift T with a positive weight sequence $(a_j)_{j\in\mathbb{Z}}$ is the weighted translation $T_{\delta_1, w*\delta_1}$ on $\ell^p(\mathbb{Z})$ with weight $w(j) = a_j$. Indeed,

$$\begin{aligned} T_{1,w*\delta_1} e_j(x) &= (w*\delta_1)(x)\cdot(f*\delta_1)(x) \\ &= \sum_{y\in\mathbb{Z}} w(x-y)\delta_1(y)\cdot\sum_{z\in\mathbb{Z}} e_j(x-z)\delta_1(z) \\ &= w(x-1)\cdot e_j(x-1) \\ &= w(j)\cdot e_{j+1}(x). \end{aligned}$$

Hence

$$T_{\delta_1, w*\delta_1} e_j = a_j \cdot e_{j+1}.$$

- The translation operator on the Lebesgue space $L^p(G)$

- The translation operator on the Orlicz space $L^\Phi(G)$

Definition (Young functions)

A continuous, convex and even function $\Phi : \mathbb{R} \to \mathbb{R}$ is called a *Young function* if it satisfies $\Phi(0) = 0$, $\Phi(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$.

Example

Both

$$\Phi(t) = \frac{|t|^p}{p} \quad (1 \leq p < \infty)$$

and

$$\Phi(t) = |t|^\alpha(1 + |\log|t||) \quad (\alpha > 1)$$

are Young functions.

- For a Young function Φ, the complementary function Ψ of Φ is given by

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\} \qquad (y \in \mathbb{R}),$$

which is also a Young function.

- If Ψ is the complementary function of Φ, then Φ is the complementary function of Ψ, and they satisfy the Young inequality

$$xy \leq \Phi(x) + \Psi(y) \qquad (x, y \geq 0).$$

Example

Let

$$\Phi(x) = \frac{|x|^p}{p} \quad (1 < p < \infty).$$

Then the complementary function Ψ of Φ is given by

$$\Psi(y) = \frac{|y|^q}{q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In the case,

$$xy \leq \Phi(x) + \Psi(y) = \frac{x^p}{p} + \frac{y^q}{q} \qquad (x, y \geq 0).$$

Orlicz spaces

Definition

Let G be a locally compact group with identity e and a right Haar measure λ. Then the *Orlicz space* $L^\Phi(G)$ is defined by

$$L^\Phi(G) = \left\{ f : G \to \mathbb{C} : \int_G \Phi(\alpha|f|)d\lambda < \infty \text{ for some } \alpha > 0 \right\}$$

where f is a Borel measurable function.

- The Orlicz space is a Banach space under the Orlicz norm defined for $f \in L^{\Phi}(G)$ by

$$\|f\|_{\Phi} = \sup \left\{ \int_G |fv| d\lambda : \int_G \Psi(|v|) d\lambda \leq 1 \right\}.$$

- One can also define the Luxemburg norm on $L^{\Phi}(G)$ by

$$N_{\Phi}(f) = \inf \left\{ k > 0 : \int_G \Phi\left(\frac{|f|}{k}\right) d\lambda \leq 1 \right\}.$$

- It is well known that these two norms above are equivalent.

Example

Let $\Phi(t) := \frac{|t|^p}{p}$ where $1 < p < \infty$. Let G be a locally compact group. Then the Orlicz space $L^\Phi(G)$ is the Lebesgue space $L^p(G)$.

Corollary (Rao and Ren, Book 1991, Corollaries 3.4.5)

Let Φ be Δ_2-regular. Then the space $C_c(G)$ of all continuous functions on G with compact support is dense in $L^\Phi(G)$.

Theorem (Rao and Ren, Book 1991, Theorem 3.5.1)

Let G be a second countable locally compact group, and let Φ be a Δ_2-regular Young function. Then the Orlicz space $L^\Phi(G)$ is separable.

Δ_2-regular

Definition

A Young function is said to be Δ_2-*regular* if there exist a constant $M > 0$ and $t_0 \geq 0$ such that $\Phi(2t) \leq M\Phi(t)$ for $t \geq t_0$.

Example

Both Young functions

$$\Phi(t) = \frac{|t|^p}{p} \quad (1 \leq p < \infty)$$

and

$$\Phi(t) = |t|^\alpha (1 + |\log |t||) \quad (\alpha > 1)$$

are Δ_2-regular.

Weighted Orlicz spaces

- A continuous function $w : G \to (0, \infty)$ is called a *weight* on G if

$$w(xy) \leq w(x)w(y) \qquad (x, y \in G).$$

- One can define the weighted Orlicz space by

$$L_w^\Phi(G) := \{f : fw \in L^\Phi(G)\}.$$

- The norm

$$\|f\|_{\Phi,w} := \|fw\|_\Phi \qquad (f \in L_w^\Phi(G))$$

is called a *weighted Orlicz norm*.

- $L_w^\Phi(G)$ is a Banach space with respect to the norm $\|\cdot\|_{\Phi,w}$.

Translation operators on $L^\Phi_w(G)$

- Let $a \in G$ and δ_a be the unit point mass at a.

- A *translation operator* T_a on $L^\Phi_w(G)$ is defined by

$$(T_a f)(x) = (f * \delta_a)(x) = \int_{y \in G} f(xy^{-1})\delta_a(y) = f(xa^{-1})$$

where $x \in G, f \in L^\Phi_w(G)$.

Theorem (Chen, Öztop and Tabatabaie, Complex Anal. Oper. Theory 2020)

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function. Given some $N \geq 2$, let $a_l \in G$ be aperiodic, and let T_{a_l} be a translation on $L^\Phi_w(G)$ for $1 \leq l \leq N$. Then we have (ii) $\Rightarrow$ (i).

(i) $T_{a_1}, T_{a_2}, \cdots, T_{a_N}$ *are disjoint topologically transitive on* $L^\Phi_w(G)$.

Theorem (continued)

(ii) *For each compact subset $K \subseteq G$ with $\lambda(K) > 0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that*

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{K\setminus E_k} |v(x)|w(x)d\lambda(x) = 0,$$

for $1 \le l \le N$,

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{E_k} |v(xa_l^{\pm n_k})|w(xa_l^{\pm n_k})d\lambda(x) = 0$$

and for $1 \le l, s \le N$ with $l \neq s$,

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{E_k} |v(xa_s^{-n_k}a_l^{n_k})|w(xa_s^{-n_k}a_l^{n_k})d\lambda(x) = 0$$

where Ω is the set of all Borel functions v on G satisfying $\int_G \Psi(|v|)d\lambda \le 1$.

Corollary

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function. Given some $N \geq 2$, let $a_l \in G$ be aperiodic, and let T_{a_l} be a translation on $L_w^\Phi(G)$ for $1 \leq l \leq N$. Then we have (ii) $\Rightarrow$ (i).

(i) $T_{a_1}, T_{a_2}, \cdots, T_{a_N}$ *are disjoint topologically mixing on* $L_w^\Phi(G)$.

Corollary (continued)

(ii) *For each compact subset $K \subseteq G$ with $\lambda(K) > 0$, there exists a sequence of Borel sets (E_n) in K such that*

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{K\setminus E_n} |v(x)|w(x)d\lambda(x) = 0,$$

for $1 \leq l \leq N$,

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{E_n} |v(xa_l^{\pm n})|w(xa_l^{\pm n})d\lambda(x) = 0$$

and for $1 \leq l, s \leq N$ with $l \neq s$,

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{E_n} |v(xa_s^{-n}a_l^n)|w(xa_s^{-n}a_l^n)d\lambda(x) = 0$$

where Ω is the set of all Borel functions v on G satisfying $\int_G \Psi(|v|)d\lambda \leq 1$.

Lemma (Chen and Chu, PAMS 2011)

An element a in a locally compact group G is aperiodic if, and only if, for each compact subset $K \subset G$, there exists $N \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > N$.

- In many familiar non-discrete groups, including the additive group $\mathbb{R}^n$, the Heisenberg group and the affine group, all elements except the identity are aperiodic.

- In the following result, we need one more assumption that

$$K \cap K a_s^{-n} a_l^n = \emptyset$$

holds for $l \neq s$ and $n \in \mathbb{N}$ large enough.

- It is easy to see this property naturally holds for some groups. Indeed, let $G = \mathbb{R}$ or $\mathbb{Z}$. Then the element $a_s^{-1} a_l \in G$ is aperiodic. Hence for n large enough,

$$\emptyset = K \cap K(a_s^{-1} a_l)^n = K \cap K a_s^{-n} a_l^n.$$

Theorem

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function. Given some $N \geq 2$ and for each compact subset $K \subset G$, let $a_l \in G$ be aperiodic for $1 \leq l \leq N$, and satisfy $K \cap K a_s^{-n} a_l^n = \emptyset$ for $l \neq s$ and $n \in \mathbb{N}$ large enough. Let T_{a_l} be a translation on $L_w^\Phi(G)$. Then we have (i) $\Rightarrow$ (ii).

(i) $T_{a_1}, T_{a_2}, \cdots, T_{a_N}$ *are disjoint topologically transitive on* $L_w^\Phi(G)$.

Theorem (continued)

(ii) *For each compact subset $K \subseteq G$ with $\lambda(K) > 0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that*

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{K\setminus E_k} |v(x)|w(x)d\lambda(x) = 0,$$

for $1 \leq l \leq N$,

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{E_k} |v(xa_l^{\pm n_k})|w(xa_l^{\pm n_k})d\lambda(x) = 0$$

and for $1 \leq l, s \leq N$ with $l \neq s$,

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{E_k} |v(xa_s^{-n_k} a_l^{n_k})|w(xa_s^{-n_k} a_l^{n_k})d\lambda(x) = 0$$

where Ω is the set of all Borel functions v on G satisfying $\int_G \Psi(|v|)d\lambda \leq 1$.

Corollary

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function. Given some $N \geq 2$ and for each compact subset $K \subset G$, let $a_l \in G$ be aperiodic for $1 \leq l \leq N$, and satisfy $K \cap K a_s^{-n} a_l^n = \emptyset$ for $l \neq s$ and $n \in \mathbb{N}$ large enough. Let T_{a_l} be a translation on $L_w^\Phi(G)$. Then we have (i) $\Rightarrow$ (ii).

(i) $T_{a_1}, T_{a_2}, \cdots, T_{a_N}$ *are disjoint topologically mixing on* $L_w^\Phi(G)$.

Corollary (continued)

(ii) *For each compact subset* $K \subseteq G$ *with* $\lambda(K) > 0$*, there exists a sequence of Borel sets* (E_n) *in* K *such that*

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{K\setminus E_n} |v(x)|w(x)d\lambda(x) = 0,$$

for $1 \leq l \leq N$*,*

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{E_n} |v(xa_l^{\pm n})|w(xa_l^{\pm n})d\lambda(x) = 0$$

and for $1 \leq l, s \leq N$ *with* $l \neq s$*,*

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{E_n} |v(xa_s^{-n}a_l^n)|w(xa_s^{-n}a_l^n)d\lambda(x) = 0$$

where Ω *is the set of all Borel functions* v *on* G *satisfying* $\int_G \Psi(|v|)d\lambda \leq 1$.

Example

Let $G = \mathbb{R}$ and $a_1 = 2, a_2 = 5$. Let w be a weight on $\mathbb{R}$. Then the translations T_2 and T_5 on $L^\Phi_w(\mathbb{R})$ are defined respectively by

$$T_2 f(x) = f(x-2) \text{ and } T_5 f(x) = f(x-5) \qquad (f \in L^\Phi_w(\mathbb{R})).$$

By the results above, operators T_2 and T_5 are disjoint topologically transitive if, and only if, given a compact subset $K \subseteq \mathbb{R}$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that

Example (continued)

$$\lim_{k\to\infty}\sup_{v\in\Omega}\int_{K\setminus E_k}|v(x)|w(x)dx=0,$$

$$\lim_{k\to\infty}\sup_{v\in\Omega}\int_{E_k}|v(x\pm 2n_k)|w(x\pm 2n_k)dx=0,$$

$$\lim_{k\to\infty}\sup_{v\in\Omega}\int_{E_k}|v(x\pm 5n_k)|w(x\pm 5n_k)dx=0$$

and

$$\lim_{k\to\infty}\sup_{v\in\Omega}\int_{E_k}|v(x\pm 3n_k)|w(x\pm 3n_k)dx=0$$

where Ω is the set of all Borel functions v on $\mathbb{R}$ satisfying $\int_{\mathbb{R}}\Psi(|v(x)|)dx\leq 1$.

Special case I

- Let T_{a_l} be generated by the power of an aperiodic element $a \in G$. That is, $a_l := a^{r_l}$ for $r_l \in \mathbb{N}$ and $1 \leq r_1 < r_2 < \cdots < r_N$.

- In this case, we can remove this assumption of the property $K \cap K a_s^{-n} a_l^n = \emptyset$. Indeed, if a is aperiodic, then for n large enough,

$$K \cap K a_s^{-n} a_l^n = K \cap K a^{-r_s n} a^{r_l n} = K \cap K a^{(r_l - r_s)n} = \emptyset$$

follows automatically.

Theorem

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function, and let $a \in G$ be aperiodic. Given $r_l \in \mathbb{N}$ with $1 \leq r_1 < r_2 < \cdots < r_N$, let $T_{a^{r_l}}$ be a translation on $L^{\Phi}_w(G)$ for $1 \leq l \leq N$. Then the following conditions are equivalent.

(i) $T_{a^{r_1}}, T_{a^{r_2}}, \cdots, T_{a^{r_N}}$ *are disjoint topologically transitive on* $L^{\Phi}_w(G)$.

Theorem (continued)

(ii) *For each compact subset $K \subseteq G$ with $\lambda(K) > 0$, there exist a sequence of Borel sets (E_k) in K and a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that*

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{K\setminus E_k} |v(x)|w(x)d\lambda(x) = 0,$$

for $1 \leq l \leq N$,

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{E_k} |v(xa^{\pm r_l n_k})|w(xa^{\pm r_l n_k})d\lambda(x) = 0$$

and for $1 \leq l, s \leq N$ with $l \neq s$,

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{E_k} |v(xa^{(r_l - r_s)n_k})|w(xa^{(r_l - r_s)n_k})d\lambda(x) = 0$$

where Ω is the set of all Borel functions v on G satisfying $\int_G \Psi(|v|)d\lambda \leq 1$.

Corollary

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function. Let $a \in G$ be aperiodic, and let T_{a^l} be a translation on $L_w^\Phi(G)$ for $1 \leq l \leq N$. Then the following conditions are equivalent.

(i) $T_a, T_{a^2}, \cdots, T_{a^N}$ *are disjoint topologically transitive on* $L_w^\Phi(G)$.

Corollary (continued)

(ii) *For each compact subset* $K \subseteq G$ *with* $\lambda(K) > 0$, *there exist a sequence of Borel sets* (E_k) *in* K *and a strictly increasing sequence* $(n_k) \subset \mathbb{N}$ *such that*

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{K\setminus E_k} |v(x)|w(x)d\lambda(x) = 0,$$

and for $1 \leq l \leq N$,

$$\lim_{k\to\infty} \sup_{v\in\Omega} \int_{E_k} |v(xa^{\pm ln_k})|w(xa^{\pm ln_k})d\lambda(x) = 0$$

where Ω *is the set of all Borel functions* v *on* G *satisfying* $\int_G \Psi(|v|)d\lambda \leq 1$.

Corollary

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function. Let $a \in G$ be aperiodic, and let T_{a^l} be a translation on $L_w^{\Phi}(G)$ for $1 \leq l \leq N$. Then $T_a, T_{a^2}, \cdots, T_{a^N}$ are disjoint topologically transitive if, and only if, $T_a \oplus T_{a^2} \oplus \cdots \oplus T_{a^N}$ is topologically transitive.

Theorem (Bès, Martin, Peris and Shkarin, JFA 2012, Theorem 3.4)

Let T be a continuous linear operator on a Banach space X. If T satisfies the Kitai Criterion, then $T, T^2, \cdots, T^N$ are disjoint topologically mixing for each $N \in \mathbb{N}$.

- An operator T on a Banach space X is said to *satisfy the Kitai Criterion* if there exist dense subsets X_0 and Y_0 of X and a map $S : Y_0 \to Y_0$ such that $TSy = y$, $T^n x \to 0$ and $S^n y \to 0$ for each $y \in Y_0$ and $x \in X_0$.

Proposition

Let G be a second countable locally compact group, and let w be a weight on G. Let Φ be a Δ_2-regular Young function, and let $a \in G$ be aperiodic. Let T_a be a translation on $L^{\Phi}_{w}(G)$. Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

(i) *For each compact subset $K \subseteq G$ with $\lambda(K) > 0$, there exists a sequence of Borel sets (E_n) in K such that*

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{K\setminus E_n} |v(x)|w(x)d\lambda(x) = 0$$

and

$$\lim_{n\to\infty} \sup_{v\in\Omega} \int_{E_n} |v(xa^{\pm n})|w(xa^{\pm n})d\lambda(x) = 0$$

where Ω is the set of all Borel functions v on G satisfying $\int_G \Psi(|v|)d\lambda \leq 1$.

Proposition (continued)

(ii) T_a *satisfies the Kitai Criterion.*

(iii) T_a *is topologically mixing on* $L^{\Phi}_{w}(G)$.

(iv) $T_{a^1}, T_{a^2}, \cdots, T_{a^N}$ *are disjoint topologically mixing.*

References

- K.-G. Grosse-Erdmann and A. Peris, Linear Chaos, Universitext, Springer, 2011.
- F. Bayart and É. Matheron, Dynamics of linear operators, Cambridge Tracts in Math. **179**, Cambridge University Press, Cambridge, 2009.
- C-H. Chu, Matrix convolution operators on groups, Lecture Notes in Math. **1956**, Springer-Verlag, Heidelberg, 2008.
- G.B. Folland, A course in abstract harmonic analysis, CRC Press, Boca Raton, 1995.
- M. M. Rao and Z. D. Ren, Theory of Orlicz spaces, Monogr. Textbooks Pure Appl. Math., **146**, Dekker, New York, 1991.

Thank you so much!